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STUDYING THE CONFINED MEMBRANE BUNDLE BY SOLVING THE SCHRÖDINGER-LIKE EQUATION USING THE BETHE ANSATZ METHOD

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Abstract: Elastic membranes stretched between two substrates adopt shapes determined by their separation length. This behavior is analogous to many biological structures, such as coiled DNA and mitochondria in eukaryotic cells. Such systems can be modeled as a stack of parallel membranes separated by layers of water. In this study, we focus on a simplified system of four membranes, where the upper and lower membranes are modeled as two rigid walls. The two-dimensional shapes of the elastic membranes confined between these walls are investigated by introducing the Morse Double Potential (MDP). In the first step, we established a Schrödinger-like equation within the continuum limit approximation. Solving this equation by using the the Bethe Ansatz methode allowed us to distinguish between the excited states and the fundamental (ground) state. Our results demonstrate that the shape of the substrate potential plays a crucial role in determining both the membrane's unbinding transition temperature and the separation length between the two confined membranes.

Keywords: Membrane, Double Morse Potential, unbinding transition, Bethe ansatz method.

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Introduction

Inside the eukaryotic cell, 98% of the membrane is located in the interior, while only 2% constitutes the outer cell membrane [1]. The stacking of these membrane sheets forms bundles within the mitochondrion. The lamellar phases formed by lipids consist of large stacks of lipid bilayers. In most studies, these membranes are assumed to be in a fluid state, and the membrane bundles are described as smectic liquid crystals. These liquid crystals exist in equilibrium with excess water. The organization of the membrane sheets within the cell has prompted many researchers to investigate the phenomenon of membrane confinement. Although the organelle exhibits three-dimensional complexity, the use of the Monge representation in studying its statistical mechanics indicates that a two-dimensional model can effectively capture its mechanical behavior.

The primary goals of the confinement study were to demonstrate that the membrane exhibits roughness as an interface governed by surface tension [2,3]. Studying the unbinding transition between two confined membranes relies on their separation distance. A theoretical study focuses on the statistical physics and dynamics of a supported bilayer biomembrane. The primary objective of this work is to develop model systems for biological membranes. Among the physical systems considered, some membranes can float freely, while the substrate is masked by the lower bilayer [4]. As a result of this study, the fluctuation-induced repulsion between the membranes, as well as between the lower membrane and the wall, is explicitly taken into account. Previous studies have employed a Gaussian variational approach, but this work demonstrates that thermal membrane fluctuations significantly influence the effective adsorption strength and the average distance to the substrate. These physical quantities exhibit a strong dependence on temperature and the membrane's elastic moduli, including the bending rigidity and the microscopic surface tension. Additionally, the authors investigate the stability domain of these supported membranes. They reveal a complex dependence on the bare adsorption forces, highlighting the intricate interplay between thermal fluctuations, membrane elasticity, and substrate interactions. Another study has investigated a two-dimensional membrane subjected to an external one-dimensional potential [5]. This research focused on the thermal fluctuations of the membrane at temperatures significantly lower than its unbinding transition temperature. As a result, the authors determined the average position of the membrane and its fluctuation amplitude. The influence of membrane confinement extends beyond determining the average position and fluctuation amplitude; it also significantly impacts the dynamics of phase separation in mixed lipid membranes [6]. Specifically, confinement affects the kinetic ordering process during phase separation. Additionally, confinement alters various physical properties of membranes. For instance, in a lamellar phase of bilayer membranes confined between two walls and subjected to a steady shear, the formation of onion-

like structures has been observed [7]. Recent work has focused on studying a stretched membrane confined between two walls [8]. The authors demonstrated that the membrane adopts a specific shape determined by its length and the volume of fluid it encloses. This study highlights the interplay between geometric constraints and fluid-structure interactions in defining the membrane's configuration. In this investigation, the authors employed a repulsive confinement pressure to prevent the membrane from intersecting the walls of the confining box. They utilized continuation methods to compute the response diagram, which allowed them to identify the specific mechanical properties of the membrane that give rise to mitochondria-like shapes. This approach highlights the role of membrane mechanics in shaping biological structures.

In practical applications, confined membranes have been characterized using modern spectroscopic techniques [9 – 12]. These techniques provide valuable insights into the structural and dynamic properties of membranes under confinement, further highlighting the importance of confinement effects in membrane studies.

The aim of this paper is to quantify the effect of confinement on the statics of a fluid membrane. In this investigation, we assume that the interaction experienced by a point on the confined membranes is similar to the interaction experienced by the hydrogen atom in the O—H—O molecule, which is part of the chromic acid (CrO_2H). To achieve this, we solved the Schrödinger-type equation exactly using the Bethe ansatz approach.

The frame of this paper is as follows: In Section 2, a detailed description of the model and basic equations. Section 3, is devoted to studying the statistics of the confined membrane. Finally, in Section 4 the conclusions are drawn.

Model and basic equations

Description of the system

From a practical standpoint, it is important to first understand the field application of the confinement membrane. The intermembrane space surrounds the bundle membrane, particularly in mitochondria (see Fig. 1).

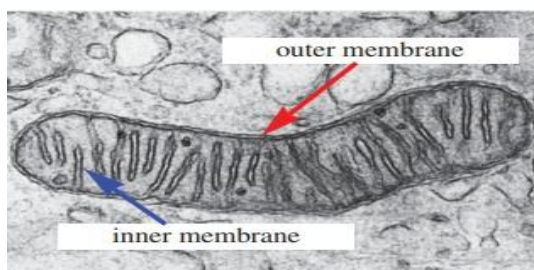


Figure 1: An experimental image of a typical mitochondrion reveals a smaller outer membrane enclosing a larger, confined inner membrane.

To simplify, we consider two membranes confined between two others, which are separated by an average distance L and are parallel to a reference plane. The bundle is

immersed in a three-dimensional euclidean space \mathbb{R}^3 filled with fluid (see Fig. 2). These two membranes interact with each other and with the two adjacent membranes. In this two-membrane approximation, which is of the mean-field type, the two membranes are confined by the other membranes located at the edges, so that, the upper well is in the position $\langle h_\alpha \rangle = \langle h_a \rangle - l_a$ while the lower wall is in the position $\langle h_\beta \rangle = \langle h_b \rangle - l_b$. The two membranes α and β , are replaced by rigid walls (Fig. 2) separated by a distance $L = \langle h_\beta \rangle - \langle h_\alpha \rangle$. Previously, it was shown that the two-membrane interactions $V(h_a - h_\alpha)$ and $V(h_b - h_\beta)$ approximated by the normalized interactions $V_\alpha^R(h_a - \langle h_\alpha \rangle)$ and by $V_\beta^R(h_b - \langle h_\beta \rangle)$, respectively [1].

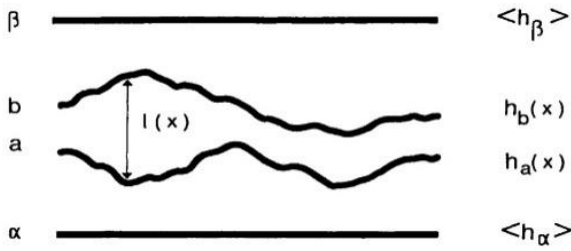


Figure2: In the two-membrane approximation for a bundle of membranes, the outer membranes α and β are replaced by rigid walls positioned at $\langle h_\alpha \rangle$ and $\langle h_\beta \rangle$, respectively [13].

The interaction energies have been determined by expanding the $V_\alpha^R(h_a - \langle h_\alpha \rangle)$ and by $V_\beta^R(h_b - \langle h_\beta \rangle)$ in powers of $h_b - \langle h_\beta \rangle$ and $h_a - \langle h_\alpha \rangle$ respectively. For small fluctuation amplitudes, this leads to the following relationship

$$V_\alpha^R(h_a - \langle h_\alpha \rangle) - V_\beta^R(h_b - \langle h_\beta \rangle) \approx V_\alpha^R(l_a) - V_\beta^R(l_b). \quad (1)$$

In the 1970s years of the last century, Canham, Helfrich, and Evans [14, 15] highlighted the crucial role played by curvature elasticity. When thermal fluctuations of the membrane are small, the Canham-Helfrich energy is given by the following relation

$$\mathcal{H}_0[h(\mathbf{x})] = \frac{1}{2} \int_S dh(\mathbf{x}) \left[\kappa (\Delta h(\mathbf{x}))^2 - \mu h^2(\mathbf{x}) \right], \quad (2)$$

here, the parameter ($\kappa \approx 20 - 50 k_B T$) for a lipid bilayer), describes the bending rigidity $\nabla = \partial e_x + \partial e_y$ is the gradient operator, and S is the projected membrane area. Due to the symmetry of the membrane, the spontaneous curvature and the Gaussian energy contribution are omitted. In our situation, the amiltonian depends explicitly on the h_a and h_b . Therefore, the relative displacement field $l = h_a - h_b$ is governed by

$$\mathcal{H}_0[l(\mathbf{x})] = \frac{1}{2} \int_S dl(\mathbf{x}) \left[\kappa (\Delta l(\mathbf{x}))^2 - \mu l^2(\mathbf{x}) \right], \quad (2a)$$

where the effective bending rigidity $\kappa = \kappa_1 \kappa_2 / (\kappa_1 + \kappa_2)$. The technique of reducing a stack of membranes to two membranes was introduced in references [16 – 18]. The harmonic potential, $\mu l^2/2$ represents the interaction

between two membranes [7]. From equation (2a), we can treat the two membranes as a single membrane confined between two walls, where l is the separation distance between the two membranes.

Confinement of a single membrane

Before describing the interactions between the membrane and the two walls, let us first examine the original interactions (i.e., interactions between membranes and substrates). Several contributions arise in the interaction potential: at short distances, volume interactions and hydrophobic or hydration repulsion dominate [19]. At intermediate distances, screened electrostatic and van der Waals interactions become attractive. To clarify the role of these interactions, we assume that the Debye screening length is small. Consequently, the potential between a flat membrane and the wall can be expressed as the sum of two terms: a decreasing exponential repulsive part that accounts for both hydration and steric contributions [20]

$$V_{\text{hyd}} = P_{\text{hyd}} e^{-z/\lambda_{\text{hyd}}}, \quad (3)$$

note that, the values of hydration pressure P_{hyd} ($1 - 10 \times 10^{-2} \text{ J.m}^{-2}$) and length λ_{hyd} ($0.1 - 0.3 \text{ nm}$) are not precisely known, and a van der Waals attractive part between the membrane and walls [21]

$$V_{\text{vdW}} = -\frac{A_H}{12\pi} \left[\frac{1}{z^2} - \frac{1}{(z+a)^2} \right], \quad (4)$$

where a is the membrane thickness, and A_H is the Hamaker's constant between interfaces lipid/water and substrate/water ($A_H \approx 1 - 10^{-21} \text{ J}$).

In order to study the confined membrane, we choose to model the potential using a simpler form: the Morse potential. This potential is characterized by only two parameters, its depth D_0 and its range η . It is given by the following relation

$$V_{\text{Morse}}(l) = D_0 [e^{-2\eta(L-l)} - 2e^{-\eta(L-l)}]. \quad (5)$$

This potential aligns with the two essential characteristics described above: attraction at intermediate-length scales and repulsion at short distances (see Fig. 3).

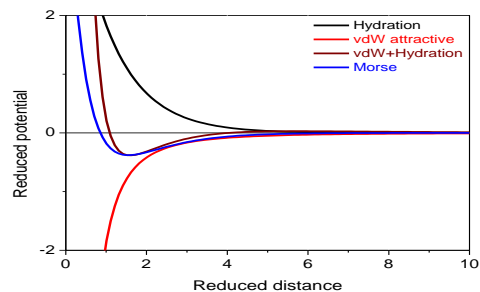


Figure 3: Substrate-membrane potential as a function of the distance to the substrate [4]

Since we are focusing on the interaction between confining walls and membranes, the impact of confinement can be modelled by incorporating a Morse potential. Each side of the membrane interacts with the wall via V_{Morse} . Thus, the total potential can be expressed using the following average formulation $V_{\text{Tot}} = [V_{\text{Morse}}(L + l) + V_{\text{Morse}}(L - l)]/2$ [6,22].

$$V_{\text{Morse}}(L + l) = D_0[e^{-2\eta(L+l)} - 2e^{-\eta(L+l)}], \quad (6)$$

$$V_{\text{Morse}}(L - l) = D_0[e^{-2\eta(L-l)} - 2e^{-\eta(L-l)}]. \quad (7)$$

Therefore, starting from the Double Morse Potential (DMP), we obtain the following total potential

$$V_{\text{Tot}}(l) = D \left[\frac{A^2}{2} \cosh(2\eta l) - 2A \cosh(\eta l) \right], \quad (8)$$

where, $A = 2e^{-\eta L}$ and $D = SD_0/2$ (where S is the membrane's surface area, introduced to ensure the depth dimension is in Joules), this potential admits two minima if and only if $A < 1$. The positions of these minima are located at:

$$l = l_{\pm} \eta^{-1} \ln \zeta, \quad \zeta = \frac{1 \pm \sqrt{1-A^2}}{A}. \quad (9)$$

The condition $\ln \zeta \geq 0$ leads to the determination of a characteristic separation distance, L^* , given by $L^* = \eta^{-1} \ln 2$. This potential is illustrated in Fig 4.

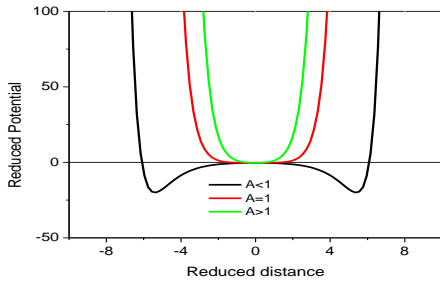


Figure 4: Double Morse Potential reduced versus reduced distance.

The statistical study a confined membrane

In this section, we will examine the role of thermal fluctuations in the behavior of confined membranes. Specifically, we will explore how the presence of walls influences the statistical properties of biological membranes. The starting point for our analysis is the partition function of the system, denoted as $Z[l]$, which describes the system's configurations and can be expressed in the following form

$$Z[l] = \int D \exp[-\beta f(l)], \quad (10)$$

here, $f = \mathcal{H} + V_{\text{Tot}}$ represents the potential energy in l , which consists of the hamiltonian \mathcal{H} and the total potential V_{Tot} . By utilizing the eigenfunctions and eigenvalues of the transfer-integral operator, the integral can be solved in the thermodynamic limit [23 – 25]

$$\int dl_{i-1} \varphi_k(l_{i-1}) e^{-\beta f(l_i, l_{i-1})} = e^{-\beta \epsilon_k} \varphi_k(l_i), \quad (10a)$$

where ϵ_k and φ_k are the eigenvalues and corresponding eigenstates, respectively. Here, $\beta = 1/k_B T$, where T is the temperature and k_B is Boltzmann's constant. Krumhansl and Schrieffer applied the statistical mechanics of the ϕ^4 field [26], and found that $z \approx \exp(-\beta \epsilon_0)$, where ϵ_0 is the lowest eigenvalue of the operator. In the continuum limit approximation, the transfer-integral eigenvalue problem can be reduced to a Schrödinger-like equation (see Appendix A).

$$-\frac{a^2}{2\beta \kappa_{\text{eff}}} \frac{d^2 \varphi(l)}{dl^2} + (\epsilon_k - s_0 + V_{\text{Tot}}(l)) \varphi(l) = 0. \quad (11a)$$

The eigenvalues ϵ_k are ordered such that $\epsilon_0 \leq \epsilon_1 \leq \epsilon_2 \leq \dots$, and the wavefunction $\varphi_k(l)$ satisfies the boundary conditions $\varphi_k(\pm\infty) = 0$. As is well-known for equations of this type, the following transformation is applied

$$\varphi_k(z) = e^{-Az\sqrt{\gamma}} \psi(z); \quad z = \cosh(\eta l), \quad (12)$$

to extract the appropriate asymptotic behavior. This leads to the following differential equation (see Appendix B)

$$(z^2 - 1)\psi''(z) + [-2A\sqrt{\gamma}z^2 + z + 2A\sqrt{\gamma}]\psi'(z) + [Az(2\gamma - \sqrt{\gamma})]\psi(z) = \left(\frac{\gamma A^2}{2} - \xi\right)\psi(z), \quad (13)$$

Where

$$\gamma = \frac{2\beta \kappa_{\text{eff}} D}{a^2 \eta^2}, \quad (14a)$$

$$\xi = \frac{2\beta \kappa_{\text{eff}} E_k}{a^2 \eta^2}, \quad (14b)$$

where, $\kappa_{\text{eff}} = \kappa_{\text{eff}} \beta$. The above equation (Eq. 13) is nearly exactly solvable, with the solution taking the form $\psi(z) = \prod_{i=1}^n (z - z_i)$, (15)

where z_i are the roots of the polynomials. These roots are determined by applying the functional Bethe ansatz method [27], leading to the following equation

$$(z^2 - 1) \sum_{i=1}^n \frac{1}{z - z_i} \sum_{i \neq j} \frac{2}{z_i - z_j} + [-2A\sqrt{\gamma}z^2 + z + 2A\sqrt{\gamma}] \sum_{i=1}^n \frac{1}{z - z_i} + A\sqrt{\gamma}z(2\sqrt{\gamma} - 1) = \left(\frac{\gamma A^2}{2} - \xi\right). \quad (16)$$

Using the expressions for $X(z)$, $Y(z)$, and $W(z)$ (as defined in Appendix B), and by comparing them with those in Eq. 15, we obtain the following relations in terms of the parameters of our problem

$$X(z) = (z^2 - 1) \quad (16a)$$

$$Y(z) = -2A\sqrt{\gamma}z^2 + z + 2A\sqrt{\gamma}, \quad (16b)$$

$$W(z) = A\sqrt{\gamma}(2\sqrt{\gamma} - 1)z - \left(\frac{\gamma A^2}{2} - \xi\right). \quad (16c)$$

Therefore, we can determine the coefficients of each polynomial, yielding the following results

$$x_0 = -1, x_1 = 0, x_2 = 1, x_3 = 0, x_4 = 0, \quad (17a)$$

$$y_0 = 2A\sqrt{\gamma}, y_1 = 1, y_2 = -2A\sqrt{\gamma}, y_3 = 0, \quad (17b)$$

$$w_0 = \xi - \frac{\gamma A^2}{2}, w_1 = A\sqrt{\gamma}(2\sqrt{\gamma} - 1), w_2 = 0. \quad (17c)$$

By using Eqs. (B.7) – (B.10) and (16a) – (16c), this substitution leads to the following relation

$$\sqrt{\gamma} = \left(n + \frac{1}{2}\right). \quad (18)$$

From this relation, we can conclude that the number of bound states is determined by the parameters of the problem. Furthermore, the exact energy eigenvalues can also be derived as

$$\xi_n = 2A\sqrt{\gamma} \sum_{i=1}^n z_i + \frac{\gamma A^2}{2} - n^2, \quad (19a)$$

along with the corresponding wavefunction

$$\varphi_n(z) = e^{-Az\sqrt{\gamma}} \prod_{i=1}^n (z - z_i). \quad (19b)$$

To determine the roots z_i , we use the equation (B.10)

$$\sum_{i \neq j} \frac{2}{z_i - z_j} = \frac{1A\sqrt{\gamma}z_i^2 - z_i - 2A\sqrt{\gamma}}{z_i^2}. \quad (19c)$$

Membranes confined between two symmetric substrates can only occupy specific energy states. To understand their behavior and the effects of confinement, the key focus of this study is to quantify the energy levels of the system.

Ground State $n = 0$

The ground state corresponds to the fundamental energy of the confined membrane, providing critical insights into the behavior of the physical system. Specifically, for the ground state (where $n = 0$), we have $\gamma = 1/2$ which allows us to find a finite temperature

$$T_0^u = \frac{2\sqrt{2D\kappa_{\text{eff}}}}{a\eta k_B}. \quad (20)$$

We observe that the temperature depends on the membrane thickness, bending rigidity, and potential depth. This temperature corresponds to the unbinding transition temperature, [1]. The expression is consistent with the results obtained using the variational method, which corresponds to a flat membrane [4]. Additionally, this relation aligns with findings for strings and DNA denaturation, which are derived from explicit transfer matrix calculations and supported by Monte Carlo simulations for membranes [27 – 29]. Using the Eqs. 18a and 18b, we obtain the following eigenvalue and corresponding wave function

$$E_0 = \frac{\eta^2 \gamma A^2}{2\beta \kappa_{\text{eff}}}, \quad \varphi_0(z) \approx \exp(-Az\sqrt{\gamma}), \quad (21)$$

where, $E_0 = \epsilon_0 - s_0$ (here ϵ_0 is the lowest eigenvalue corresponding to the energy of the ground state), the wave function has no nodes and can therefore be used to describe the ground state of the system.

The quantity that provides information about the separation of the two membranes is their average separation, $\langle l \rangle$. The standard scaling arguments imply that $\langle l \rangle \sim \langle z \rangle$, for which we use the relation [28]

$$\langle z \rangle = \langle \varphi_0 | z | \varphi_0 \rangle = \int z \varphi_0^2(z) dz. \quad (22)$$

In the limit of large N , the result is dominated by the lowest eigenvalue ϵ_0 associated with the eigenfunction $\varphi_0(z)$, leading to the following relation

$$\langle l \rangle \sim \langle z \rangle = 2A^2 \left[\left(\frac{T}{T_0^u} \right)^2 \sinh \left(A\hat{L} \frac{T_0^u}{T} \right) - A\hat{L} \frac{T}{T_0^u} \cosh \left(A\hat{L} \frac{T_0^u}{T} \right) \right]. \quad (23)$$

In the case where the minimum positions $l = l_{\pm}$ satisfy the condition $\ln \zeta > 0$, the average separation between the two membranes takes the following form

$$\langle l \rangle^* \sim 2A^2 \left[\left(\left(\frac{T}{T_0^u} \right)^2 - A\ln 2 \right) 2^{A\frac{T_0^u}{T}-1} - \left(\left(\frac{T}{T_0^u} \right)^2 + A\ln 2 \right) 2^{-A\frac{T_0^u}{T}-1} \right]. \quad (23a)$$

Note that the mean separation (Eq. 23) is exact and is shown in Fig. 5, illustrating the thermal evolution of $\langle l \rangle$. From the curve, we observe that above the unbinding temperature $T > T_0^u$, the mean separation between the membranes is substantial, indicating that the membranes are fully separated. Conversely, below the unbinding

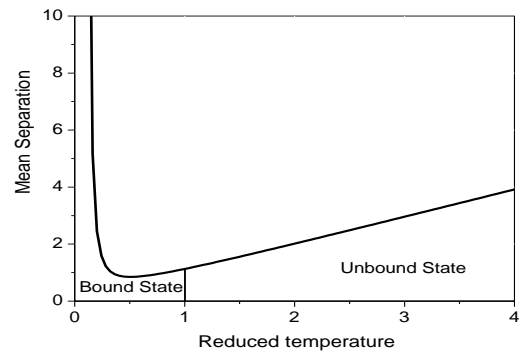


Figure 5: Mean separation as a function of reduced temperature.

temperature $T < T_0^u$, the membranes remain bound. The asymptotic behavior of the average separation is described by two distinct regimes, (i) for $T/T_0^u \rightarrow \infty$, the average separation follows:

$$\lim_{T/T_0^u \rightarrow \infty} \langle l \rangle \sim \frac{T}{T_0^u} \left(\frac{T}{T_0^u} - 1 \right) e^{\frac{\hat{L}T}{T_0^u}}, \quad (24)$$

indicating an exponential growth in separation as the temperature increases significantly above the unbinding temperature, (ii) $T/T_0^u \rightarrow 0$, the average separation follows

$$\lim_{T/T_0^u \rightarrow 0} \langle l \rangle \sim \hat{L} \left(\frac{T}{T_0^u} \right)^3 \left[1 - \frac{\hat{L}}{2} - \left(\frac{T}{T_0^u} \right)^{-2} \right], \quad (25)$$

showing a power-law dependence as the temperature approaches zero, consistent with the membranes remaining bound at low temperatures.

Excited States Cases $n = 1, n = 2$ and $n = 3$

In quantum physics, the excited states of a system are quantum states with energies higher than the ground state. The ground state represents the lowest energy level, and any state with energy above it is considered an excited state. By analogy, we can apply this concept to our system. For the first excited state ($n = 1$), Eqs. 17, 18a and 18b yields

$$E_1 = \frac{(\eta k_B T)^2}{2\kappa_{\text{eff}}} \left[2A\sqrt{\gamma}z_1 + \frac{\gamma A^2}{2} - 1 \right], \quad (26a)$$

$$\varphi_1(z) = (z - z_1)e^{-Az\sqrt{\gamma}}, \quad (26b)$$

$$T_1^u = \frac{1}{3}T_0^u. \quad (26c)$$

Using the condition for z_1 derived from relationship (18c), we deduce the following expression

$$z_1 = \frac{1}{4A\sqrt{\gamma}}(1 \pm \sqrt{1 + 16A^2\gamma}). \quad (26c)$$

Similarly, for $n = 2$, corresponding to the second excited state, we obtain the following relations

$$E_2 = \frac{(\eta k_B T)^2}{2\kappa_{\text{eff}}} \left[2A\sqrt{\gamma}(z_1 + z_2) + \frac{\gamma A^2}{2} - 4 \right], \quad (27a)$$

$$\varphi_2(z) = (z - z_1)(z - z_2)e^{-Az\sqrt{\gamma}}, \quad (27b)$$

$$T_2^u = \frac{1}{5}T_0^u. \quad (27c)$$

Provided z_1 and z_2 satisfy the equations (using relationship 18c)

$$\frac{2}{z_1 - z_2} = \frac{A\sqrt{\gamma}z_1^2 - z_1 - A\sqrt{\gamma}}{z_1^2 - 1}, \quad (27d)$$

$$\frac{2}{z_2 - z_1} = \frac{A\sqrt{\gamma}z_2^2 - z_2 - A\sqrt{\gamma}}{z_2^2 - 1}. \quad (27e)$$

In the last case, for $n = 3$, corresponding to the third excited state, we derive the following relations

$$E_3 = \frac{(\eta k_B T)^2}{2\kappa_{\text{eff}}} \left[2A\sqrt{\gamma}(z_1 + z_2 + z_3) + \frac{\gamma A^2}{2} - 9 \right], \quad (28a)$$

$$\varphi_3(z) = (z - z_1)(z - z_2)(z - z_3)e^{-Az\sqrt{\gamma}}, \quad (28b)$$

$$T_3^u = \frac{1}{7}T_0^u, \quad (28c)$$

provided z_1 , z_2 and z_3 satisfy the equations (we use the relationship 18c)

$$\frac{2}{z_1 - z_2} + \frac{2}{z_1 - z_3} = \frac{A\sqrt{\gamma}z_1^2 - z_1 - A\sqrt{\gamma}}{z_1^2 - 1}, \quad (28d)$$

$$\frac{2}{z_2 - z_1} + \frac{2}{z_2 - z_3} = \frac{A\sqrt{\gamma}z_2^2 - z_2 - A\sqrt{\gamma}}{z_2^2 - 1}, \quad (28e)$$

$$\frac{2}{z_3 - z_1} + \frac{2}{z_3 - z_2} = \frac{A\sqrt{\gamma}z_3^2 - z_3 - A\sqrt{\gamma}}{z_3^2 - 1}, \quad (28f)$$

Noted that $T_n^u = T_0^u/(2n + 1)$ ($n \neq 0$), represents the temperature of each excited state. We observe that, for well-defined problem parameters, each state of the membrane can be quantified.

From Eqs. 21, 26a, 27a, and 28a, we can determine the energy gap ΔE between the ground state and the unbound states. This energy gap represents the dissipated energy when the membrane transitions from the unbound state to the ground state if $\Delta E < 0$. Conversely, if $\Delta E > 0$, the membrane gains energy from an external source to transition from the ground state to the unbound state.

$$\frac{E_1 - E_0}{k_B T} = \frac{\eta^2}{2} \left(\frac{T}{\theta} \right) \left[A \left(\frac{T_u}{T} \right) z_1 - 1 \right], \quad (29a)$$

$$\frac{E_2 - E_0}{k_B T} = \frac{\eta^2}{2} \left(\frac{T}{\theta} \right) \left[A \left(\frac{T_u}{T} \right) (z_1 + z_2) - 4 \right], \quad (29b)$$

$$\frac{E_3 - E_0}{k_B T} = \frac{\eta^2}{2} \left(\frac{T}{\theta} \right) \left[A \left(\frac{T_u}{T} \right) (z_1 + z_2 + z_3) - 9 \right], \quad (29c)$$

$$\frac{E_2 - E_1}{k_B T} = \frac{\eta^2}{2} \left(\frac{T}{\theta} \right) \left[A \left(\frac{T_u}{T} \right) z_2 - 3 \right], \quad (29d)$$

$$\frac{E_3 - E_1}{k_B T} = \frac{\eta^2}{2} \left(\frac{T}{\theta} \right) \left[A \left(\frac{T_u}{T} \right) (z_2 + z_3) - 8 \right], \quad (29e)$$

$$\frac{E_3 - E_2}{k_B T} = \frac{\eta^2}{2} \left(\frac{T}{\theta} \right) \left[A \left(\frac{T_u}{T} \right) z_3 - 5 \right], \quad (29f)$$

Where $\theta = \kappa_{\text{eff}}/k_B$ is a characteristic temperature.

Conclusions and Remarks

In this investigation, we focused on membrane bundles, which play a crucial role in living cells. For this study, we limited our analysis to a bundle consisting of only four membranes, where the upper and lower membranes were modeled as walls. This problem has attracted the attention of several investigators. In our approach, we utilized the Bethe ansatz method to solve the Schrödinger-like equation. Initially, we assumed that the membranes are of the DMP type, which leads to unbinding or adhesion transitions between the bound and unbound states of the membranes. By applying the continuum limit approximation, we derived a Schrödinger-like equation. Its solutions were found to consist of bound states and unbound states, with each state corresponding to a specific eigenfunction. The energy levels take discrete values, indicating that the levels are quantized. In the thermodynamic limit of a large number N , the level energies are dominated by the lowest eigenvalue ϵ_0 . In this regime, the two membranes undergo a continuous unbinding transition at a critical temperature T_c , referred to as the unbinding temperature, $T_0^u = 2\sqrt{2\kappa_{\text{eff}}D}/a\eta k_B$. Strikingly, we find that the length separation is continuous near the unbinding temperature. This observation appears to contradict the established literature, where $T \rightarrow T_0^u$, [28]. In the other hand, we demonstrate also, that each unbound state ($n \neq 0$) corresponds to a discrete temperature T_n^u , which decreases as the quantum number n increases, following the relation $T_n^u = T_0^u/(2n + 1)$.

On the other hand, we have demonstrated that energy levels are quantized, and the energy gap is directly related to the energy gained or dissipated by the membrane.

Appendix A

In this appendix, we attempt to develop the Eq. 9a by using the continuum limit approximation method, the first step is to make the following substitution (we use: $(\nabla^2 l)^2 = (l_{i-1} + 2l_i + l_{i+1})^2 \simeq (l_{i-1} - l_i)^2$).

$$\frac{\kappa}{2} \int d^2\mathbf{x} (\nabla^2 l)^2 \rightarrow \frac{\kappa}{a^2} \sum_{i=1}^n (l_{i-1} - l_i)^2, \quad A.1$$

$$\frac{\mu}{2} \int d^2\mathbf{x} (\nabla^2 l)^2 \rightarrow \frac{\mu a^2}{2} \sum_{i=1}^n (l_{i-1} - l_i)^2, \quad A.2$$

where, a is the thickness of the membrane. Therefore, the total Hamiltonian of the $f(l_i, l_{i-1}) = \mathcal{H} + V_{\text{Tot}}$ which can be rewritten

$$f(l_i, l_{i-1}) = \frac{\kappa_{\text{eff}}}{a^2} \sum_{i=1}^n (l_{i-1} - l_i)^2 + D \left[\frac{A^2}{2} \text{csch}(2\eta l) - 2 \cosh(\eta l) \right], \quad A.3$$

where $\kappa_{\text{eff}} = \kappa + \mu a^2 S$. if we use the following notation system is $\hat{\kappa} = \beta \kappa_{\text{eff}}$. A gradient expansion of (9a) involves

$$\varphi_k(l_{i-1}) = \varphi_k(l_i) + (l_{i-1} - l_i) \frac{d\varphi_k(l_i)}{dl_i} + \frac{1}{2!} (l_{i-1} - l_i)^2 \frac{d^2\varphi_k}{dl_i^2} + \dots, \quad A.4$$

and performing the Gaussian integrals over l_i . The integration is done between $-L$ and $+L$, we get the following Schrödinger-like equation

$$-\frac{a^2}{2\beta\hat{\kappa}} \frac{d^2\varphi_k(l)}{dl^2} + (E_k + V_{\text{Tot}})\varphi_k(l) = 0, \quad A.5$$

where,

$$S_0 = k_B T \ln \left[\sqrt{\frac{\pi}{\hat{\kappa}}} \text{erf}(\hat{L}\sqrt{\hat{\kappa}}) \right], \quad A.6$$

is entropic, and $E_k = \epsilon_k - s_0$. Note that $\hat{L} = \eta L$ is a dimensionless length.

Appendix B

In this appendix, we rederive the solution of Eq. 12, the first step is to make the substitution (Eq. 11), we insert these in Eq. 10 we obtain

$$(z^2 - 1)\psi''(z) + [-2A\sqrt{\gamma}z^2 + z + 2A\sqrt{\gamma}]\psi'(z) + \left[A^2 \left(\gamma - \frac{2\beta\hat{\kappa}D}{\eta^2} \right) z^2 + A \left(-\sqrt{\gamma} + \frac{4\beta\hat{\kappa}D}{\eta^2} \right) z - \gamma A^2 + \frac{2\beta\hat{\kappa}E_k}{\eta^2} \right] \psi(z) = 0. \quad B.1$$

In the following, we will hereafter call the γ factor permits to quantify the energy levels, due to the ambiguity in the determination of the values D and η , therefore we have

$$\gamma = \frac{2\beta\hat{\kappa}D}{\eta^2}, \quad B.2$$

and the constant dimensionless given by

$$\xi = \frac{2\beta\hat{\kappa}E_k}{\eta^2}. \quad B.3$$

With these considerations, we get the following relation

$$(z^2 - 1)\psi''(z) + [-2A\sqrt{\gamma}z^2 + z + 2A\sqrt{\gamma}]\psi'(z) + [+Az(-\sqrt{\gamma} + 2\gamma)]\psi(z) = \left(\frac{\gamma A^2}{2} - \xi \right) \psi(z). \quad B.4$$

This equation can be rewritten as follows

$$\left[X(z) \frac{d^2}{dz^2} + Y(z) \frac{d}{dz} + W(z) \right] \psi(z) = 0, \quad B.5$$

Where, $X(z) = \sum_{k=0}^4 x_k z^k$, $Y(z) = \sum_{k=0}^3 y_k z^k$ and $W(z) = \sum_{k=0}^2 w_k z^k$ are the polynomes, with x_k, y_k and w_k are the coefficients. Since the differential equation (B.5) has degree n polynomial solution $\psi(z) = \prod_{i=1}^n (z - z_i)$, we get the new form [27]

$$X(z) \sum_{i=1}^n \frac{1}{z - z_i} \sum_{i \neq j} \frac{1}{z_i - z_j} + Y(z) \sum_{i=1}^n \frac{1}{z - z_i} + W(z) = 0. \quad B.6$$

Given a pair of polynomials $X(z)$ and $Y(z)$, then the values of the coefficients w_k of polynomial $W(z)$, with distinct roots z_1, z_2, \dots, z_n are given by the following relations [29]

$$w_0 = -[y_3 + 2(n-1)x_4] \sum_{i=1}^n z_i^2 - 2x_4 \sum_{i < j} z_i z_j - [2(n-1)x_3 + y_2] \sum_{i=1}^n z_i - n[(n-1)x_2 + y_1], \quad B.7$$

$$w_1 = -[2(n-1)x_4 + y_3] \sum_{i=1}^n z_i - n(n-1)x_3 - ny_2 \quad B.8$$

$$w_2 = -n(n-1)x_4 - ny_3. \quad B.9$$

Note that the roots z_1, z_2, \dots, z_n satisfy the Bethe ansatz following equations

$$\sum_{i \neq j}^n \frac{1}{z_i - z_j} + \frac{y_3 z_i^3 + y_2 z_i^2 + y_1 z_i + y_0}{x_4 z_i^4 + x_3 z_i^3 + x_2 z_i^2 + x_1 z_i + x_0} = 0. \quad B.10$$

From the Eqs. B.7 – B.10, we can deduce all polynomials $W(z)$ such that the differential Eq. (B.5) has degree n polynomial solutions (B.6).

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